Notes to the perfect Euler brick problem

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April 10, 2022

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1 Basic Discussion

Problem 1. The *perfect Euler brick problem* is an open Diophantine problem that asks if there exists a rational point $P(x,y,z) \neq (0,0,0)$ in the 3-dimensional Euclidean space such that the distance $\sqrt{x^2+y^2}, \sqrt{y^2+z^2}, \sqrt{z^2+x^2}, \sqrt{x^2+y^2+z^2}$ are all rational numbers.

Definition 1. 1) Let $X \subset \mathbb{P}^6 = \mathrm{Proj}\mathbb{Z}[x,y,z,r_1,r_2,r_3,r_4]$ be the projective variety defined by the homogeneous ideal $I(X) =< x^2+y^2-r_1^2, y^2+z^2-r_2^2, z^2+x^2-r_3^2, x^2+y^2+z^2-r_4^2 >$. Define $f : X(\mathbb{R}) \to \mathbb{RP}^2, p(x:y:z:r_1:r_2:r_3:r_4) \mapsto (x:y:z)$, and denote $q_1(0:0:1), q_2(1:0:0), q_3(0:1:0)$.

2) Let $X_\epsilon \subset \mathbb{P}^6 = \mathrm{Proj}\mathbb{Z}[x,y,r_1,r_2,r_3,r_4,z]$ be the projective variety defined by the homogeneous ideal $I(X) =< x^2+y^2+\epsilon r_4^2-r_1^2, y^2+z^2+\epsilon r_4^2-r_2^2, z^2+x^2+\epsilon r_4^2-r_3^2, x^2+y^2+z^2-r_4^2 >$. Define $f_\epsilon : X_\epsilon(\mathbb{R}) \to \mathbb{RP}^2, p(x:y:z:r_1:r_2:r_3:r_4) \mapsto (x:y:z)$,

2 Topological inspection of $X(\mathbb{R})$ and $X_\epsilon(\mathbb{R})$

$$p_1(0:0:\pm1:0:\pm1:\pm1:1), \quad p_2(\pm1:0:0:\pm1:0:\pm1:1), \quad p_3(0:\pm1:0:\pm1:\pm1:0:1),$$

For $\epsilon > 0$, $X_\epsilon(\mathbb{R})$ is the disjoint union of 8 spheres. We give its connected components decomposition as $X_\epsilon(\mathbb{R}) = \bigsqcup_{0\leq i\leq 7} X_{\epsilon,i}$. Define a map $g_1 : X_\epsilon(\mathbb{R}) \to \mathbb{R}^3$, for any $p(x:y:r_1:r_2:r_3:r_4:z) \in X_\epsilon(\mathbb{R}), g(p) = (\frac{x}{r_4}, \frac{y}{r_4}, \frac{z}{r_4})$. For any $t \in \mathbb{R}_\times, g(tp) = (\frac{tx}{tr_4}, \frac{ty}{tr_4}, \frac{tz}{tr_4}) = g(p)$, so g is well defined. The image of g is $g(X_\epsilon(\mathbb{R})) = \mathbb{S}^2 = \{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$. For a given point $s_1(x,y,z) \in \mathbb{S}^2$, its fiber is given by $g^{-1}(s_1) = \{(x:y:z:\pm\sqrt{x^2+y^2+\epsilon} : \pm\sqrt{y^2+z^2+\epsilon} : \pm\sqrt{z^2+x^2+\epsilon} : 1)\}$. We conclude that $g : X_\epsilon(\mathbb{R}) \to \mathbb{S}^2 \subset \mathbb{R}^3\backslash\{0\}$ is an 8-fold covering, and its locally a smooth diffeomorphism everywhere on $X_\epsilon(\mathbb{R})$. The exact correspondence between $X_{\epsilon,i}, 0 \leq i \leq 7$ and the signs of $(\frac{r_1}{r_4}, \frac{r_2}{r_4}, \frac{r_3}{r_4})$ are given in the following table:

i	$\mathrm{sgn}(\frac{r_1}{r_4})$	$\mathrm{sgn}(\frac{r_2}{r_4})$	$\mathrm{sgn}(\frac{r_3}{r_4})$
0	+	+	+
1	-	+	+
2	+	-	+
3	-	-	+
4	+	+	-
5	-	+	-
6	+	-	-
7	-	-	-

Table 1: Correspondence between $X_{\epsilon,i}$ and the signs of $(\frac{r_2}{r_1}, \frac{r_3}{r_1}, \frac{r_4}{r_1})$

$$\begin{aligned}
p_{1,0,0}(0:0:1:\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{1,0,1}(0:0:-1:\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{1,1,0}(0:0:1:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{1,1,1}(0:0:-1:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{1,2,0}(0:0:1:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{1,2,1}(0:0:-1:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{1,3,0}(0:0:1:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{1,3,1}(0:0:-1:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{1,4,0}(0:0:1:\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{1,4,1}(0:0:-1:\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1),\\
p_{1,5,0}(0:0:1:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{1,5,1}(0:0:-1:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1),\\
p_{1,6,0}(0:0:1:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{1,6,1}(0:0:-1:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1),\\
p_{1,7,0}(0:0:1:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{1,7,1}(0:0:-1:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:1).
\end{aligned}$$

$$\begin{aligned}
p_{1,0}(0:0:1:0:1:1:1), &\quad p_{1,1}(0:0:-1:0:1:1:1),\\
p_{1,2}(0:0:1:0:-1:1:1), &\quad p_{1,3}(0:0:-1:0:-1:1:1),\\
p_{1,4}(0:0:1:0:1:-1:1), &\quad p_{1,5}(0:0:-1:0:1:-1:1),\\
p_{1,6}(0:0:1:0:-1:-1:1), &\quad p_{1,7}(0:0:-1:0:-1:-1:1).
\end{aligned}$$

$$\begin{aligned}
p_{2,0,0}(1:0:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{2,0,1}(-1:0:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{2,1,0}(1:0:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{2,1,1}(-1:0:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{2,2,0}(1:0:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{2,2,1}(-1:0:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{2,3,0}(1:0:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), &\quad p_{2,3,1}(-1:0:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
p_{2,4,0}(1:0:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{2,4,1}(-1:0:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1),\\
p_{2,5,0}(1:0:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{2,5,1}(-1:0:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1),\\
p_{2,6,0}(1:0:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{2,6,1}(-1:0:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1),\\
p_{2,7,0}(1:0:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), &\quad p_{2,7,1}(-1:0:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1).
\end{aligned}$$

$$\begin{aligned}
p_{2,0}(1:0:0:1:0:1:1), &\quad p_{2,1}(-1:0:0:1:0:1:1),\\
p_{2,2}(1:0:0:-1:0:1:1), &\quad p_{2,3}(-1:0:0:-1:0:1:1),\\
p_{2,4}(1:0:0:1:0:-1:1), &\quad p_{2,5}(-1:0:0:1:0:-1:1),\\
p_{2,6}(1:0:0:-1:0:-1:1), &\quad p_{2,7}(-1:0:0:-1:0:-1:1).
\end{aligned}$$

$$
\begin{aligned}
p_{3,0,0}(0:1:0:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{3,0,1}(0:-1:0:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1),\\
p_{3,1,0}(0:1:0:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{3,1,1}(0:-1:0:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1),\\
p_{3,2,0}(0:1:0:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{3,2,1}(0:-1:0:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1),\\
p_{3,3,0}(0:1:0:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{3,3,1}(0:-1:0:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1),\\
p_{3,4,0}(0:1:0:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1), &\quad p_{3,4,1}(0:-1:0:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\
p_{3,5,0}(0:1:0:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1), &\quad p_{3,5,1}(0:-1:0:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\
p_{3,6,0}(0:1:0:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1), &\quad p_{3,6,1}(0:-1:0:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\
p_{3,7,0}(0:1:0:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1), &\quad p_{3,7,1}(0:-1:0:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),
\end{aligned}
$$

$$
\begin{aligned}
p_{3,0}(0:1:0:1:1:0:1), &\quad p_{3,1}(0:-1:0:1:1:0:1),\\
p_{3,2}(0:1:0:-1:1:0:1), &\quad p_{3,3}(0:-1:0:-1:1:0:1),\\
p_{3,4}(0:1:0:1:-1:0:1), &\quad p_{3,5}(0:-1:0:1:-1:0:1),\\
p_{3,6}(0:1:0:-1:-1:0:1), &\quad p_{3,7}(0:-1:0:-1:-1:0:1).
\end{aligned}
$$

Proposition 1. 1) When $\epsilon > 0$, the homology groups of $X_\epsilon(\mathbb{R})$ are:

$$H_2(X_\epsilon(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}^8, \quad H_2(X_\epsilon(\mathbb{R}),\mathbb{Z}) = 0, \quad H_2(X_\epsilon(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}^8.$$

2) When $\epsilon = 0$, the homology groups of $X_0(\mathbb{R}) = X(\mathbb{R})$ are:

$$H_2(X(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}^8, \quad H_1(X(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}^{17}, \quad H_0(X(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}.$$

3) When $\epsilon < 0$, the homology groups of $X_\epsilon(\mathbb{R})$ are:

$$H_2(X_\epsilon(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}, \quad H_2(X_\epsilon(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}^{34}, \quad H_2(X_\epsilon(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}.$$

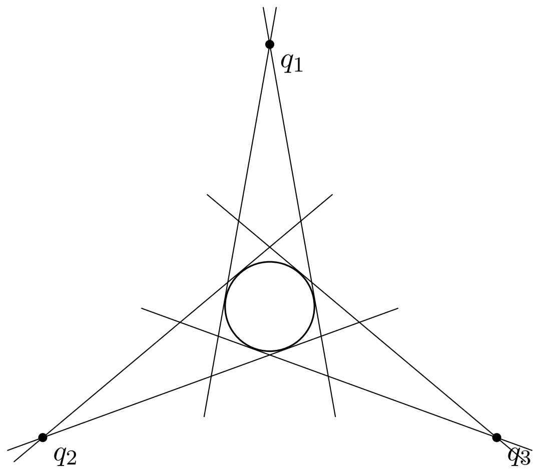
3 Topological inspection of $X(\mathbb{C})$ and $X_\epsilon(\mathbb{C})$

Denote four conic curves and their irreducible components in $\mathbb{CP}^2$ by

$$
\begin{aligned}
s_1: x^2+y^2=0, &\quad l_{1,0}: x+iy=0, \quad l_{1,1}: x-iy=0,\\
s_2: y^2+z^2=0, &\quad l_{2,0}: y+iz=0, \quad l_{2,1}: y-iz=0,\\
s_3: z^2+x^2=0, &\quad l_{3,0}: z+ix=0, \quad l_{3,1}: z-ix=0,\\
&s_4: x^2+y^2+z^2=0.
\end{aligned}
$$

The 6 projective complex lines above has 15 joints in total, all of them meets 2 lines, among which 3 of them belong to single conic curve listed above. The last circle is tangent to each of the 6 lines above. The coordinates of these points are given as follows:

$$
\begin{aligned}
&q_{\mathrm{db},1,2}(1:i:1), \quad q_{\mathrm{db},1,3}(1:-i:1), \quad q_{\mathrm{db},1,4}(1:i:-1), \quad q_{\mathrm{db},1,4}(1:-i:-1),\\
&q_{\mathrm{db},2,1}(1:1:i), \quad q_{\mathrm{db},2,3}(1:1:-i), \quad q_{\mathrm{db},2,4}(1:-1:i), \quad q_{\mathrm{db},1,4}(1:-1:-i),\\
&q_{\mathrm{db},3,1}(i:1:1), \quad q_{\mathrm{db},3,2}(-i:1:1), \quad q_{\mathrm{db},3,4}(i:1:-1), \quad q_{\mathrm{db},1,4}(-i:1:-1),\\
&\qquad q_1(0:0:1), \quad q_2(1:0:0), \quad q_3(0:1:0).
\end{aligned}
$$



Let $s = \bigcup_{1 \le i \le 3} s_i$ be the union of the three conics above. Then a disjoint union of s can be written as

$$s = \bigsqcup_{1 \le i \le 3, 0 \le j \le 1} l_{i,j} \bigsqcup_{1 \le i \le 3} q_i \bigsqcup_{1 \le i \ne j \le 4} q_{\text{db},i,j} \bigsqcup_{1 \le i \le 6} q_{\tan,i}.$$

f is 16-fold on $\mathbb{CP}^2 \backslash s$, 8-fold on s besides the 18 joints between different conics, 4-fold on 18 joints which each meets 2 lines or circle. So formally we have the following partition of $X(\mathbb{C})$ ignoring boundary maps, and its euler characteristic is known:

$$X(\mathbb{C}) = \bigsqcup 16 \times (\mathbb{CP}^2 \backslash s) \bigsqcup 8 \times (s \backslash \{q_{\text{db},*}, q_{\tan,*}\}) \bigsqcup 4 \times \{q_{\text{db},*}, q_{\tan,*}\}$$
$$\chi(s) = 6 * 2 + 2 - 15 - 6 = -7, \quad \chi(\mathbb{CP}^2 \backslash s) = 10,$$
$$\chi(s \backslash \{q_{\text{db},*}, q_{\tan,*}\}) = \chi(s) - \chi(q_{\text{db},*}, q_{\tan,*}) = -25, \quad \chi(q_{\text{db},*}, q_{\tan,*}) = 18,$$
$$\chi(X(\mathbb{C})) = 16 * 10 + 8 * (-25) + 4 * 18 = 32.$$

The 3 relations about Chern numbers and Euler characteristic of $X(\mathbb{C})$ are as follows:

$$c_1^2 + c_2 = 12 * \chi(X(\mathbb{C})) = 384, \quad 384 - c_2 = c_1^2 \le 3c_2, \quad c_2 \ge 96, \quad c_1^2 \le 288$$
$$5c_1^2 - c_2 + 36 = 6c_1^2 - 348 \ge 0, \quad c_1^2 \ge 58, \quad 8 \le c_1 \le 16.$$

Since the Kodaira dimension $\kappa(X_\epsilon(\mathbb{C})) = 2$ when $c_1^2, c_2 > 0$, we conclude that $X_\epsilon(\mathbb{C})$ is a surface of general type.

4 Rational points on X and X_ϵ

Observe that when $x = 0$ (or $y = 0$, or $z = 0$), the equation degenerate to the classical Pythagorean equation. $X(\mathbb{Q}) \cap \{x = 0\}$ is isomorphic to the rational points on a unit circle, so are $X(\mathbb{Q}) \cap \{y = 0\}$ and $X(\mathbb{Q}) \cap \{z = 0\}$.

Appendix

References

[1] Bjorn Poonen, Rational points on varieties.

[2] James Milne, Class Field Theory.

[3] Alexei Skorobogatov. Torsors and Rational Points.